CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2015 Assessment Examination

FORM VI

MATHEMATICS EXTENSION I

Monday 18th May 2015

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 112 boys

Examiner

LYL

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The polynomial $P(x) = x^4 + x^3 - 7x^2 - x + 6$ has four linear factors. Which expression below is NOT a factor of $P(x)$?

(A) $x - 1$

(B) $x - 2$

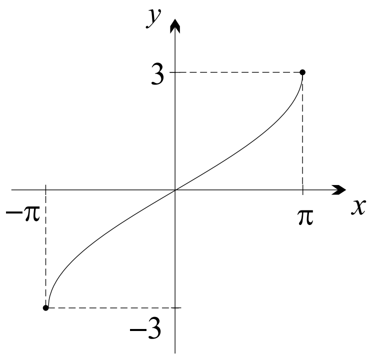
(C) $x + 1$

(D) $x + 6$

QUESTION TWO

Which graph best represents $y = 2\sin^{-1}\frac{x}{3}$?

(A)



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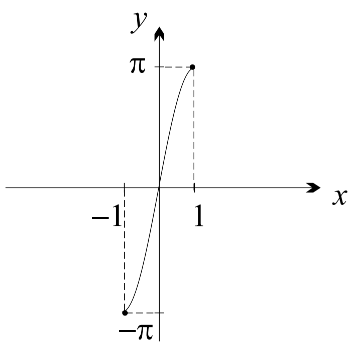

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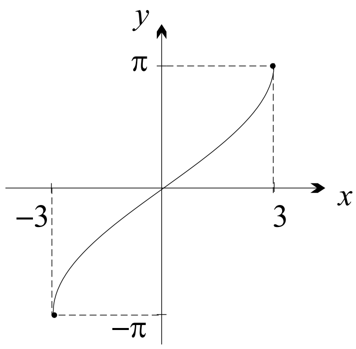

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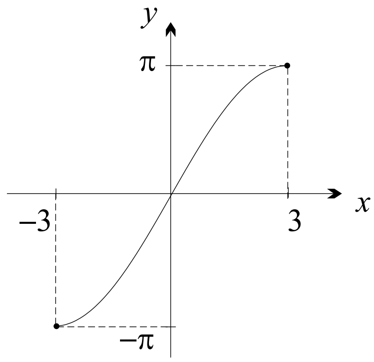

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QUESTION THREE

A particle is moving in simple harmonic motion according to the equation

$$x = 1 + 6\sin(2t + \tfrac{5\pi}{6})\,.$$

In what interval does the particle oscillate?

(A) $-6 \le x \le 6$

(B) $-5 \le x \le 7$

(C) $5 \le x \le 7$

(D) $-1 \le x \le 3$

QUESTION FOUR

The exact value of $\sec\left(\sin^{-1}(-\frac{1}{3})\right)$ is:

(A) $-\frac{3\sqrt{2}}{4}$

(B) $\frac{3\sqrt{2}}{4}$

(C) $\frac{2\sqrt{2}}{3}$

(D) $-\frac{2\sqrt{2}}{3}$

QUESTION FIVE

The expression $\sqrt{3}\sin x - \cos x$ is equivalent to:

(A) $2\sin(x - \frac{\pi}{6})$

(B) $2\sin(x + \frac{\pi}{6})$

(C) $2\sin(x + \frac{5\pi}{6})$

(D) $2\sin(x - \frac{5\pi}{6})$

Exam continues overleaf ...

QUESTION SIX

Which of the following is an expression for $\displaystyle\int \cos^2 2x\,dx$?

(A) $\dfrac{x}{2} - \dfrac{1}{8}\sin 4x + C$

(B) $x - \dfrac{1}{4}\sin 4x + C$

(C) $\dfrac{x}{2} + \dfrac{1}{8}\sin 4x + C$

(D) $x + \dfrac{1}{4}\sin 4x + C$

QUESTION SEVEN

A particle moves on a horizontal line so that its displacement x cm from the origin is given by $x = t^3 - 5t^2 - 3t + 4$. Take right as the positive direction.

At time $t = 2$ seconds the particle is:

(A) right of the origin, travelling to the left and accelerating to the right

(B) left of the origin, travelling to the left and accelerating to the right

(C) left of the origin, travelling to the right and accelerating to the right

(D) right of the origin, travelling to the right and accelerating to the left

QUESTION EIGHT

Consider the function $f(x) = \sin x + \frac{1}{2}\cos 2x$ in the interval $0 \le x \le 2\pi$. Which of the following is the x-coordinate of a stationary point of $f(x)$?

(A) 0

(B) $\frac{\pi}{6}$

(C) $\frac{7\pi}{6}$

(D) $\frac{11\pi}{6}$

Exam continues next page ...

QUESTION NINE

A projectile is fired with an initial velocity of $30\,\text{m/s}$ at an angle of elevation of 60° to the horizontal. Let x and y be the respective horizontal and vertical components of the displacement from the point of projection, and take $g = 10\,\text{m/s}^2$.

The initial conditions are:

(A) $\dot{x} = 15\sqrt{3}$, $\ddot{x} = -10$, $\dot{y} = 15$ and $\ddot{y} = 0$

(B) $\dot{x} = 30\sqrt{3}$, $\ddot{x} = 0$, $\dot{y} = 30\sqrt{3}$ and $\ddot{y} = -10$

(C) $\dot{x} = 15\sqrt{3}$, $\ddot{x} = 0$, $\dot{y} = 15$ and $\ddot{y} = -10$

(D) $\dot{x} = 15$, $\ddot{x} = 0$, $\dot{y} = 15\sqrt{3}$ and $\ddot{y} = -10$

QUESTION TEN

Find the values of x for which $(x+1)(x-2)(x-3) > 0$.

(A) $-1 < x < 2$ or $x > 3$

(B) $x < -1$ or $2 < x < 3$

(C) $x > -1$ or $2 < x < 3$

(D) $-1 < x < 3$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\displaystyle\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx$. **2**

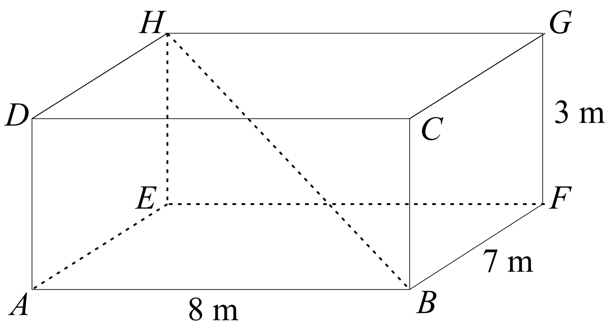
(b) Find the value of k if $x-2$ is a factor of $P(x) = x^3 - 3kx + 10$. **2**

(c) The equation $x^3 + 3x^2 - 2x + 1 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha\beta + \alpha\gamma + \beta\gamma$ and $\alpha\beta\gamma$. **2**

(ii) Hence, evaluate $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **1**

(d)



The diagram above shows a rectangular prism with dimensions 8 m by 7 m by 3 m. Find the angle the diagonal BH makes with the base $ABFE$. Leave your answer correct to the nearest minute. **2**

(e) Without the aid of calculus, sketch the graph of $P(x) = (x+1)(x-2)(x+3)^3$. Show clearly any features including all intercepts with the axes. **2**

(f) Initially a ball is thrown at 20 m/s at an angle of elevation of 30° from the top of a building 40 m high. The equations of motion of the ball are

$$x = 10t\sqrt{3}$$
$$y = -5t^2 + 10t$$

where x and y are the horizontal and vertical components of displacement from the point of projection at time t seconds after the ball is thrown.

(i) At what time will the ball hit the ground? **2**

(ii) Find the horizontal range of its flight. **1**

(iii) Find the Cartesian equation of its path. **1**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Factorise $x^3 - 5x^2 + 8x - 4$. **2**

(ii) Hence solve the equation $x^3 - 5x^2 + 8x - 4 = 0$. **1**

(b) The function $f(x) = 3 - \sqrt{x-2}$ is defined over the domain $x \geq 2$. **2**
Find the equation of the inverse function $f^{-1}(x)$ and state its domain.

(c) Prove that $\sin(\theta + \frac{\pi}{3})\sin(\theta - \frac{\pi}{3}) = \sin^2\theta - \frac{3}{4}$. **2**

(d) The acceleration of a particle P is given by $\ddot{x} = -2e^{-x}$ where x is the displacement from the origin O and right is taken as the positive direction. The particle starts at the origin with a velocity of $2\,\text{m/s}$.

(i) Show that $v^2 = 4e^{-x}$. **1**

(ii) Assuming that v is positive, find the displacement as a function of time. **2**

(iii) Briefly describe the displacement and velocity of the particle as $t \to \infty$. **1**

(iv) Explain why the velocity could be assumed to be positive in part (ii). **1**

(e) (i) Differentiate $y = x\tan^{-1}x$. **1**

(ii) Hence find a primitive of $\tan^{-1}x$. **1**

(iii) Find the area bounded by the curve $y = \tan^{-1}x$, the x-axis and the line $x = 1$. **1**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Two zeroes of a polynomial $P(x)$ of degree 3 are -2 and -4. When $x = 1$ it takes the value of 15 and when $x = -3$ it takes the value of 7. Find the polynomial. **3**

(b) The motion of a particle is given by $x = 1 + 2\sin 3t$, where $t \geq 0$.

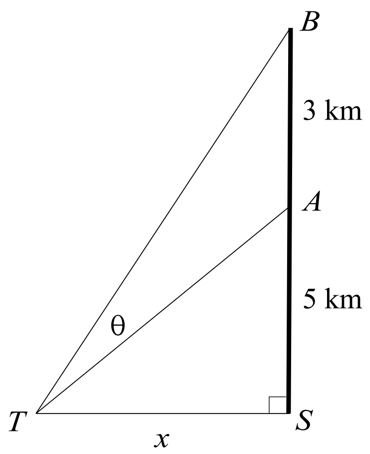
(i) Prove that the motion is simple harmonic by showing that **1**

$$\ddot{x} = -n^2(x - x_0).$$

(ii) Write down the period and the amplitude of the motion. **2**

(iii) Find the first two times when the particle returns to the centre of motion. Give your answers as exact values. **2**

(c)



The diagram above shows a technician T observing work on a pipeline that is being built out from the shores of Darwin. The technician is standing onshore x km due west of the start of the pipeline S. He can see two company boats A and B which are respectively 5 km and 8 km due north of the point S. Let θ be the angle that AB subtends at T.

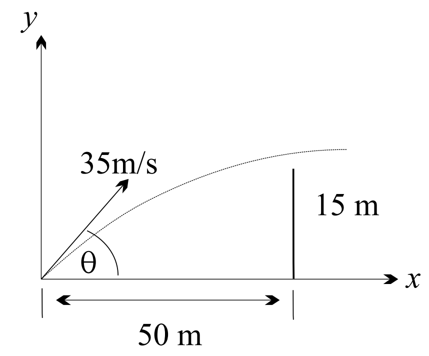
(i) Show that $\theta = \tan^{-1}\dfrac{8}{x} - \tan^{-1}\dfrac{5}{x}$. **1**

(ii) Show that θ is maximised when the technician is $2\sqrt{10}$ km from the point S. **4**

(iii) Hence show that the maximium value of θ is $\tan^{-1}\left(\dfrac{3}{4\sqrt{10}}\right)$. **2**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a stone thrown with a velocity of 35 m/s at an angle θ so that it just clears a 15 m high wall that is 50 m from the origin on the same horizontal plane.

(i) Show that the two equations of motion for the horizontal and vertical components of displacement are respectively: **2**

$$x = 35t\cos\theta$$
$$y = 35t\sin\theta - 5t^2$$

Assume that $g = 10\ \text{m/s}^2$ and t is time in seconds.

(ii) Find the angles at which the stone could be thrown. Give your answers correct to the nearest degree. **3**

(b) A line with gradient m intersects the cubic curve $y = (x-1)(x+2)(x-3)$ at the point $P(3, 0)$ and at two other points Q and R.

(i) Show that the x coordinates of the points of intersection satisfy the equation: **1**

$$x^3 - 2x^2 - (m+5)x + 6 + 3m = 0$$

(ii) Find the equation of the line through P which is also a tangent to the curve at another distinct point. **3**

(c) On a certain day the depth of water in a harbour is 3 metres at low tide and 9 metres at high tide. Low tide occurs at 5:00 am and the following high tide at 1:00 pm. Assume the rise and fall of the tides is simple harmonic. Find between what times on that day a ship may safely enter the harbour, if a minimum depth of 4 metres of water is required. **3**

(d) (i) Find the general solutions of the equation $2\sin 3x\cos 4x - 1 = \cos 4x - 2\sin 3x$. **2**

(ii) Hence find the two smallest positive solutions of this equation. **1**

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} \, e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}\,, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}\,, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\text{NOTE}: \ \ln x = \log_e x, \ x > 0$$

CANDIDATE NUMBER

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2015
Assessment Examination
FORM VI
MATHEMATICS EXTENSION I
Monday 18th May 2015

- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Form VI

Ext 1 Assessment 2015

Q11 a) $\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int_0^{\frac{1}{6}} \frac{1}{\sqrt{\frac{1}{9}-x^2}}\,dx$

$= \frac{1}{3}\left[\sin^{-1}3x\right]_0^{\frac{1}{6}}$ ✓

$= \frac{1}{3}\left(\sin^{-1}\frac{1}{2} - 0\right)$

$= \frac{\pi}{18}$ ✓

b) $P(2) = 8 - 6k + 10$ ✓ $\qquad P(2) = 0$

$18 - 6k = 0$

$-6k = -18$

$\underline{k = 3}$ ✓

c) i) $\alpha\beta + \alpha\gamma + \beta\gamma = -2$ ✓

$\alpha\beta\gamma = -1$ ✓

ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$= 2$ ✓

d) $BE^2 = AE^2 + AB^2$

$= 8^2 + 7^2$

$= 113$ ✓

$BE = \sqrt{113}$

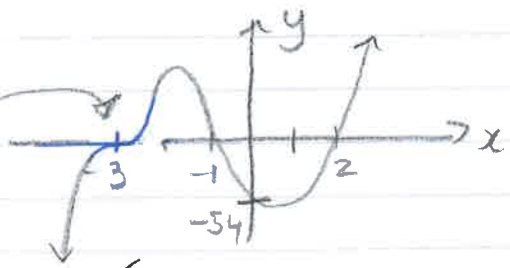
ΔHEB right angled

$\tan\theta = \frac{3}{\sqrt{113}}$

$\theta \doteqdot 15^\circ 46'$ (nearest minute) ✓

e)

$x = -3$
triple root



horizontal point of inflexion

– clear concavity change

$\left.\begin{array}{l} x = -1 \\ x = 2 \end{array}\right\}$ single roots.

intercepts ✓

shape and single/triple roots must be clearly shown

MC

Q1 D

Q2 C

Q3 B

Q4 B

Q5 A

Q6 C

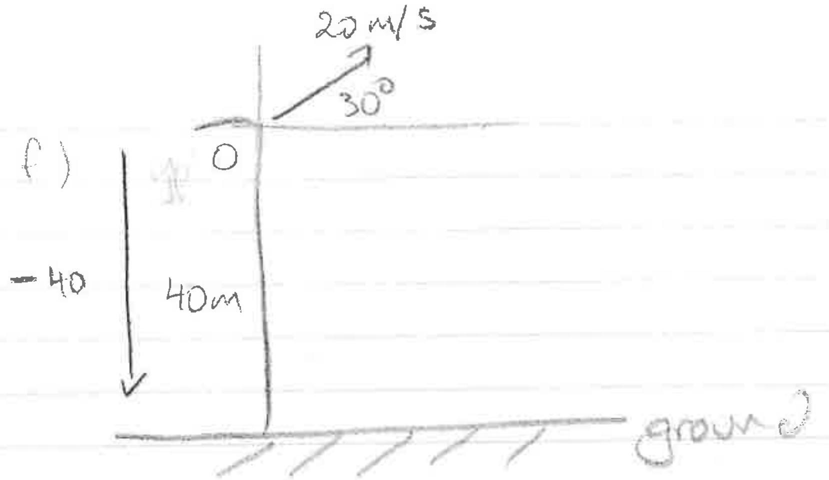
Q7 B

Q8 B

Q9 D

Q10 A

Q1 f)



i) $t = 0 \quad y = 0$

Ground is at $y = -40$

$$-5t^2 + 10t = -40$$
$$5t^2 - 10t + 40 = 0$$
$$t^2 - 2t - 8 = 0$$
$$(t+2)(t-4) = 0$$
$$t = -2 \text{ or } 4$$
$$t = 4 \text{ seconds} \quad (t > 0)$$

ii) $x = 10 \times 4 \times \sqrt{3}$
$$= 40\sqrt{3}$$

iii) $t = \dfrac{x}{10\sqrt{3}}$ $\quad y = -5\left(\dfrac{x}{10\sqrt{3}}\right)^2 + 10\left(\dfrac{x}{10\sqrt{3}}\right)$

$$= -\frac{\cancel{5}x^2}{\cancel{100}_{20} \times 3} + \frac{x}{10\sqrt{3}}$$

$$y = -\frac{x^2}{60} + \frac{x\sqrt{3}}{3}$$

$$= \frac{20x\sqrt{3} - x^2}{60}$$

Q12 a) i) let $P(x) = x^3 - 5x^2 + 8x - 4 = 0$

Test $P(1) = 1 - 5 + 8 - 4$

$= -4 + 8 - 4$

$= 0$

$P(2) = 8 - 5 \times 4 + 8 \times 2 - 4$

$= 8 - 20 + 16 - 4$

$= 0$

$(x-1)$ and $(x-2)$ are factors ✓ at least one factor

$1 + 2 + \alpha = 5$ (α is the third root)

$\alpha = 2$

$P(x) = (x-1)(x-2)^2$ ✓ long div or other methods ok

ii) $(x-1)(x-2)^2 = 0$

$x = 1$ or $x = 2$ ✓

b) $f(x) = 3 - \sqrt{x-2}$

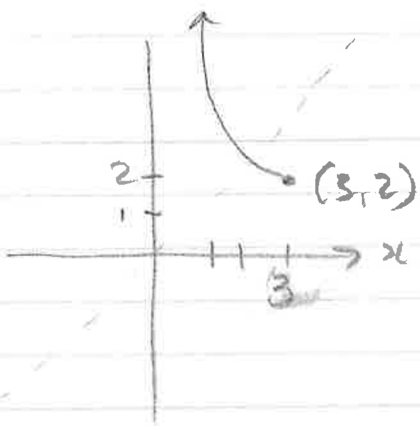
Swap x and y and rearrange

$x = 3 - \sqrt{y-2}$

$x - 3 = -\sqrt{y-2}$

$(x-3)^2 = y - 2$

$y = (x-3)^2 + 2$ ✓



Domain $x \leq 3$ ✓

c) LHS $= \sin\left(\theta + \frac{\pi}{3}\right)\sin\left(\theta - \frac{\pi}{3}\right)$

$= \left(\sin\theta\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos\theta\right)\left(\sin\theta\cos\frac{\pi}{3} - \sin\frac{\pi}{3}\cos\theta\right)$

$= \left(\frac{\sin\theta}{2} - \frac{\sqrt{3}\cos\theta}{2}\right)\left(\frac{\sin\theta}{2} - \frac{\sqrt{3}\cos\theta}{2}\right)$ ✓

$= \frac{\sin^2\theta}{4} - \frac{3\cos^2\theta}{4}$

$= \frac{\sin^2\theta - 3(1 - \sin^2\theta)}{4}$ ✓

$= \frac{4\sin^2\theta - 3}{4}$

$= \sin^2\theta - \frac{3}{4}$ as required

Q12 d) i) $\ddot{x} = -2e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\frac{1}{2}v^2 = 2e^{-x} + \frac{1}{2}C$

$v^2 = 4e^{-x} + C$

when $x=0$ $v=2$

$4 = 4e^{-0} + C$

$C = 0$

$v^2 = 4e^{-x}$

ii) At $x=0$ $v=2$ given $v>0$

$v = 2e^{-\frac{x}{2}}$

$\frac{dx}{dt} = \frac{2}{e^{\frac{x}{2}}}$

$\frac{dt}{dx} = \frac{e^{\frac{x}{2}}}{2}$

integrate

$t = \frac{1}{2} \times 2e^{\frac{x}{2}} + D$

$t = e^{\frac{x}{2}} + D$

$t=0$ $x=0$

$0 = 1 + D$

$D = -1$

$t = e^{\frac{x}{2}} - 1$

$e^{\frac{x}{2}} = t+1$

$\frac{x}{2} = \ln(t+1)$

$x = 2\ln(t+1)$

iii) The particle starts at the origin and moves right. As $t \to \infty$, $x \to \infty$, $v \to 0^+$ ie. slowing down as $t \to \infty$

iv) Initially v is positive

$t=0$ $x=0$ $v=2$

$t>0$ $x>0$ so v can never equal zero

Hence v can never be negative.

Q12 e) i) $y = x\tan^{-1}x$ $\quad u = x \quad v = \tan^{-1}x$

$= uv \quad u' = 1 \quad v' = \dfrac{1}{1+x^2}$

$y' = \dfrac{x}{1+x^2} + \tan^{-1}x$ ✓

ii) $\int \tan^{-1}x \, dx + \int \dfrac{x}{1+x^2} \, dx = x\tan^{-1}x$

$\int \tan^{-1}x \, dx = x\tan^{-1}x - \frac{1}{2}\int \dfrac{2x}{1+x^2} \, dx$ ✓

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

iii) $\int_0^1 \tan^{-1}x \, dx$

$= \left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= (\tan^{-1}1 - \frac{1}{2}\ln 2) - 0$

$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)$ ✓

$= \dfrac{\pi - 2\ln 2}{4}$ units^2

Q13 a) $P(x) = (x+2)(x+4)(ax+b)$

since $P(x)$ degree 3 a & b constants.

$P(1) = 15$

$3 \times 5 \times (a+b) = 15$ ✓

$a + b = 1$ (1)

$P(-3) = 7$

$-1 \times 1 \times (-3a+b) = 7$

$-3a + b = -7$ (2)

(1) − (2)

$4a = 8$ ✓

$\left.\begin{aligned} a &= 2 \\ b &= -1 \end{aligned}\right\}$

$P(x) = (x+2)(x+4)(2x-1)$ ✓

b) i) $x = 1 + 2\sin 3t$

$\dot{x} = 6\cos 3t$

$\ddot{x} = -18\sin 3t$ ✓

$= -9(1 + 2\sin 3t - 1)$

$= -3^2(x - 1)$

which is in the form

$= -n^2(x - x_0)$

which proves the motion is SHM

ii) period $n = 3$

$T = \frac{2\pi}{3}$ ✓

amplitude $= 2$ ✓

iii) centre of motion $x_0 = 1$

$1 = 1 + 2\sin 3t$

$2\sin 3t = 0$ ✓

$\sin 3t = 0$

$3t = \pi$ or 2π

$t = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ ✓

Q13 c) i) $\triangle ATS$ right angled

Let $\angle ATS = \alpha$

$\tan\alpha = \frac{5}{x}$

$\alpha = \tan^{-1}\frac{5}{x}$

$\triangle BTS$ right angled $\tan(\alpha + \theta) = \frac{8}{x}$ ✓

$\alpha + \theta = \tan^{-1}\frac{8}{x}$

must show some working to derive θ.

$\alpha + \theta - \alpha = \theta$

$\theta = \tan^{-1}\frac{8}{x} - \tan^{-1}\frac{5}{x}$

ii) $\theta = \tan^{-1}\frac{8}{x} - \tan^{-1}\frac{5}{x}$

$\frac{d\theta}{dx} = \frac{-8}{x^2+64} + \frac{5}{x^2+25}$ ✓

When $\frac{d\theta}{dx} = 0$

$\frac{5}{x^2+25} - \frac{8}{x^2+64} = 0$

$\frac{5(x^2+64) - 8(x^2+25)}{(x^2+25)(x^2+64)} = 0$ ✓

$5x^2 + 5\times 64 - 8x^2 - 200 = 0$

$-3x^2 + 120 = 0$

$3x^2 = 120$

$x^2 = 40$

$x = \sqrt{40}$

$= 2\sqrt{10}$ ✓ $(\doteqdot 6.32)$

$\frac{d}{dx}\tan^{-1}(8x^{-1})$

let $u = 8x^{-1}$

$\frac{du}{dx} = -8x^{-2}$

$y = \tan^{-1}u$

$\frac{dy}{du} = \frac{1}{1+u^2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{\left[1+\left(\frac{8}{x}\right)^2\right]} \times \frac{-8}{x^2}$

$= \frac{-8}{x^2+64}$

Must show maximum with table of values.

x	5	$2\sqrt{10}$	7
$\frac{d\theta}{dx}$	$\doteqdot 0.010$	0	$\doteqdot 0.0032$

$\frac{45}{89 \times 50}$ ✓

$\frac{5(7^2+64) - 8(7^2+25)}{113 \times 75}$

must provide working/values for mark.

Q13 iii) $\theta = \tan^{-1}\dfrac{8}{2\sqrt{10}} - \tan^{-1}\dfrac{5}{2\sqrt{10}}$

let $\alpha = \tan^{-1}\dfrac{8}{2\sqrt{10}}$ $\quad \beta = \tan^{-1}\dfrac{5}{2\sqrt{10}}$

$\tan\alpha = \dfrac{8}{2\sqrt{10}}$ $\quad \tan\beta = \dfrac{5}{2\sqrt{10}}$

$\theta = \alpha - \beta$ ✓

$$\tan\theta = \tan(\alpha - \beta)$$

$$= \frac{\tan\alpha - \tan\beta}{1 - \tan\alpha\tan\beta}.$$

$$= \frac{\frac{8}{2\sqrt{10}} - \frac{5}{2\sqrt{10}}}{1 + \frac{\cancel{8}\,\cancel{4}\,2}{\cancel{2}\sqrt{10}} \times \frac{5}{\cancel{2}\sqrt{10}}}$$ ✓

$$= \frac{3}{2\sqrt{10}} \times \frac{1}{2}$$

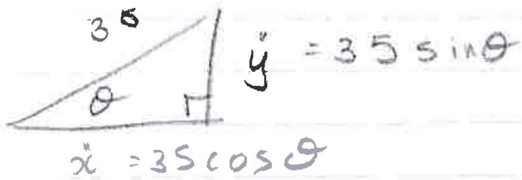
$$= \frac{3}{4\sqrt{10}}$$

so $\tan\theta = \dfrac{3}{4\sqrt{10}}$

$\theta = \tan^{-1}\dfrac{3}{4\sqrt{10}}$ as required

FORM VI ASSESSMENT 2015

Q14 a) i)



$\ddot{x} = 0$

integrate wrt x

$\dot{x} = C_1$

when $t=0$ $\dot{x} = 35\cos\theta$ $C_1 = 0$

$\dot{x} = 35\cos\theta$

integrate

$x = 35t\cos\theta + C_2$

when $t=0$ $x=0$ $C_2 = 0$

$x = 35t\cos\theta$

✓

must derive equations

$\ddot{y} = -10$

integrate wrt x

$\dot{y} = -10t + C_3$

when $t=0$ $\dot{y} = 35\sin\theta$

$C_3 = 35\sin\theta$

$\dot{y} = -10t + 35\sin\theta$

integrate wrt x

$y = \frac{-10t^2}{2} + 35t\sin\theta + C_4$

$t=0$ $y=0$ $C_4 = 0$

$y = -5t^2 + 35t\sin\theta$

✓

ii) $x=50$ $y=15$

$t = \frac{x}{35\cos\theta}$

$= \frac{\cancel{50}\,10}{\cancel{35}\,7\cos\theta}$

$= \frac{10}{7\cos\theta}$

$y = 35t\sin\theta - 5t^2$

$15 = \frac{\cancel{35}^5 \times 10\sin\theta}{\cancel{7}\cos\theta} - \frac{5\times10^2}{7^2\cos^2\theta}$ ✓

$15 = 50\tan\theta - \frac{500}{49}(1+\tan^2\theta)$

Q14 a) ii) continued

$15 \times 49 = 49 \times 50 \tan\theta - 500 - 500 \tan^2\theta$

$500\tan^2\theta - 2450\tan\theta + 1235 = 0$ ✓

$\div 5$

$100\tan^2\theta - 490\tan\theta + 247 = 0$

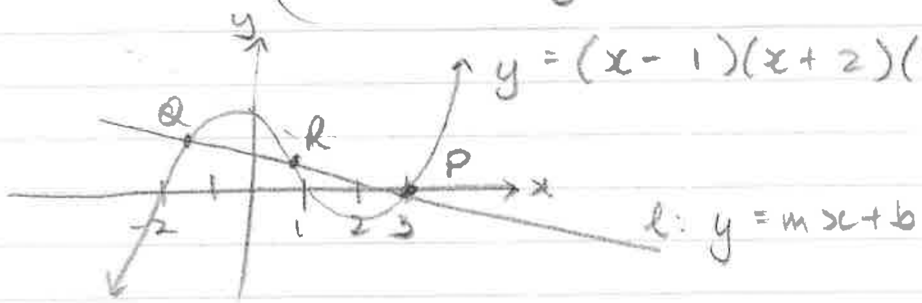
$$\tan\theta = \frac{+490 \pm \sqrt{(490)^2 - 4 \times 100 \times 247}}{200}$$ ✓

$\tan\theta \doteqdot 4.33$ $\qquad \tan\theta \doteqdot 0.571$

$\theta \doteqdot 77^\circ$ $\qquad \theta \doteqdot 30^\circ$ ✓

(nearest degree)

b)



$P(3, 0)$ lies on l so $\quad 0 = 3m + b$

$\underline{b = -3m}$

$l: y = mx - 3m$

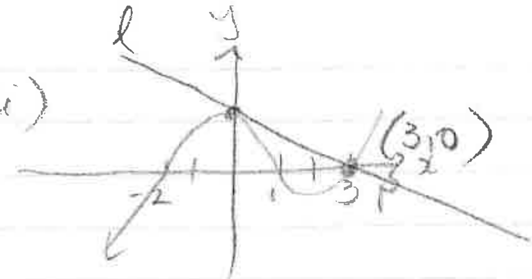
intersection of curves

$mx - 3m = x^3 - 2x^2 - 5x + 6$

$x^3 - 2x^2 - 5x - mx + 6 + 3m = 0$ ✓

$x^3 - 2x^2 - (5+m)x + 6 + 3m = 0$

Q14 b) ii)



let there be a double root α represent the tangency.

so the roots are α, α and 3

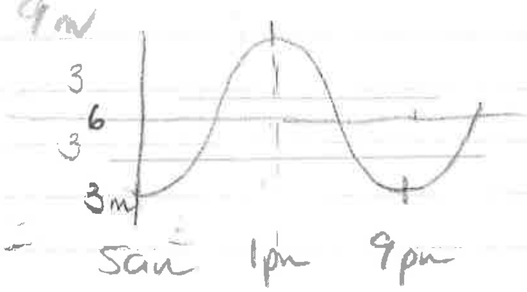
$$\alpha + \alpha + 3 = 2$$
$$2\alpha = -1$$
$$\alpha = -\frac{1}{2} \quad \checkmark$$

$$3\alpha + 3\alpha + \alpha^2 = -(m+5)$$
$$6\alpha + \alpha^2 = -(m+5)$$
$$\cancel{6}^{3} \times -\frac{1}{\cancel{2}} + \frac{1}{4} = -m-5$$
$$-2\frac{3}{4} = -m-5$$
$$m = -5 + 2\frac{3}{4} \quad \checkmark$$
$$m = -\frac{9}{4}$$

$$y = -\frac{9}{4}x - 3 \times -\frac{9}{4}$$
$$4y = -9x + 27$$
$$\underline{9x + 4y - 27 = 0} \quad \checkmark$$

1pm

Q14 c) high tide 9m



5am low tide

centre of motion 6 m

low tide to high tide = 8h

1 period is $\underline{16h}$ $\longrightarrow$ $T = \frac{2\pi}{16}$

$= \frac{\pi}{8}$

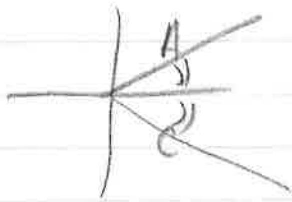
$x = 6 - 3\cos\left(\frac{\pi t}{8}\right)$ ✓

$4 = 6 - 3\cos\frac{\pi t}{8}$

$-2 = -3\cos\frac{\pi t}{8}$

$\cos\frac{\pi}{8}t = \frac{2}{3}$

$\frac{\pi t}{8} = \cos^{-1}\left(\frac{2}{3}\right)$



$\frac{\pi t}{8} \doteq 0.841$ ✓

$t \doteq \frac{0.841 \times 8}{\pi}$

$\doteq 2.14h$

$\frac{\pi t}{8} \doteq 2\pi - 0.841$

$t = \frac{(2\pi - 0.841) \times 8}{\pi}$

$\doteq 13.86h$

The ship can pass safely between $\underline{7:08am}$ and $\underline{6:51pm}$ ✓

* On that certain day, the ship could also pass safely from 12 midnight to 2:51am

Q14 d) $2\sin 3x \cos 4x - 1 - \cos 4x + 2\sin 3x = 0$

$$2\sin 3x(\cos 4x + 1) - 1(\cos 4x + 1) = 0$$ ✓

$$(2\sin 3x - 1)(\cos 4x + 1) = 0$$

If $2\sin 3x = 1$

$\sin 3x = \frac{1}{2}$

$$3x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$$

$$3x = n\pi + (-1)^n \frac{\pi}{6}$$

$$x = \frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$$ ✓

If $\cos 4x = -1$

$$4x = 2n\pi \pm \cos^{-1}(-1)$$

$$4x = 2n\pi \pm \pi$$

$$x = \frac{n\pi}{2} \pm \frac{\pi}{4}$$

$x = \frac{n\pi}{2} \pm \frac{\pi}{4}$	$x = \frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$

Looking for two smallest positive solutions

if $n = 0, 1$	if $n = 0, 1$
$x = 0 \pm \frac{\pi}{4}, \frac{\pi}{2} \pm \frac{\pi}{4}$	$x = 0 + \frac{\pi}{18}, \frac{\pi}{3} - \frac{\pi}{18}$
$x = \frac{\pi \times 18}{4 \times 18}, \cancel{\frac{3\pi}{4}}$	$x = \frac{\pi}{18}, \frac{5\pi \times 4}{18 \times 4} = \frac{20\pi}{72}$
$= \frac{18\pi}{72}$	

✓

The two smallest positive solutions are $x = \frac{\pi}{18}$ and $x = \frac{\pi}{4}$